MATHEMATICS OF COMPUTATION
VOLUME 46, NUMBER 174
APRIL 1986, PAGES 591-599

On Weighted Chebyshev-Type Quadrature Formulas

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Abstract. A weighted quadrature formula is of Chebyshev type if it has equal coefficients and real (but not necessarily distinct) nodes. For a given weight function we study the set $T(n,d)$ consisting of all Chebyshev-type formulas with n nodes and at least degree d. It is shown that in nonempty $T(n,d)$ there exist two special formulas having "extremal" properties. This result is used to prove uniqueness and further results for E-optimal Chebyshev-type formulas. For the weight function $w \equiv 1$, numerical investigations are carried out for $n \leqslant 25$.

1. Introduction. Let w be a nonnegative weight function on the interval (a,b), $-\infty \leqslant a < b \leqslant \infty$, admitting moments m_j of all order

$$m_j = \int_a^b x^j w(x)\,dx, \qquad j = 0,1,2,\ldots, \quad m_0 > 0. \tag{1.1}$$

We consider (weighted) Chebyshev-type quadrature formulas [7]. These are quadrature formulas Q_n with equal coefficients and real (but not necessarily distinct) nodes:

$$\begin{gathered} Q_n[f] := c\sum_{i=1}^{n} f(x_i), \qquad -\infty < x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n < \infty, \\ \int_a^b f(x)w(x)\,dx = Q_n[f] + R_n[f]. \end{gathered} \tag{1.2}$$

By this definition, it is possible that some nodes x_i are not contained in the interval (a,b). Q_n has at least degree d (of exactness) if

$$R_n[p_i] = 0, \qquad i = 0,1,\ldots,d, \tag{1.3}$$

where p_i, here and throughout this paper, denotes the monomial $p_i(x) := x^i$. If $d \geqslant 0$, the coefficient c in (1.2) is determined by (1.3):

$$c = m_0/n. \tag{1.4}$$

The maximal possible degree of a Chebyshev-type quadrature formula with n nodes is denoted by d_n.

Let, in the following, $T(n,d)$ be the set of all Chebyshev-type quadrature formulas with n nodes and at least degree d. One has $T(n,d+1) \subseteq T(n,d)$ and $T(n,d) \subseteq T(kn,d)$ for every $k \in \mathbf{N}$. For $n > 2$, a simple calculation shows that

Received March 26, 1984; revised November 30, 1984 and April 29. 1985.
1980 *Mathematics Subject Classification*. Primary 65D30.

each set $T(n,2)$ contains an infinite number of elements. In case of $d_n \geqslant n$ the set $T(n, d_n)$ contains only one element, the so-called "Chebyshev quadrature formula in the strict sense". Weight functions w which allow such formulas for every $n \in \mathbf{N}$ are rare [7, p. 109]. In case of $d_n < n$ the set $T(n, d_n)$ possibly contains more than one element. To select some of these formulas several criteria can be found in the literature.

From a historical point of view it may be obvious to consider such quadrature formulas $Q_n^{\text{opt}} \in T(n, d_n)$, which minimize $|R_n[p_{d_n+1}]|$ among all $Q_n \in T(n, d_n)$. Such quadrature formulas are called E-optimal [9], [2], [7]. In case of weight functions w, which are symmetric with respect to the (finite) interval (a, b), several authors have distinguished between symmetric, i.e., $x_i - a = b - x_{n-i+1}$ for $i = 1, \dots, n$, and unsymmetric formulas with regard to E-optimality [10], [9], [2]. There is computational evidence that E-optimal formulas for symmetric weight functions are indeed symmetric [7, p. 113]. Gautschi and Yanagiwara [9] have shown that symmetry would follow, if E-optimal formulas are unique in $T(n, d_n)$. One aim of this paper is to prove the uniqueness of E-optimal formulas in general.

Several authors have proposed other criteria to select special Chebyshev-type formulas—necessarily not contained in $T(n, d_n)$ (see, e.g., [7]). Therefore, it may be of interest to study the set $T(n, d)$ in general. If $T(n, d)$ contains more than one element, we show that there exists in $T(n, d)$ an infinite number of formulas which have pairwise distinct nodes. In this case, there also exists in $T(n, d)$ an infinite number of interpolatory quadrature formulas (for definition, see, e.g., [3]). Among these interpolatory quadrature formulas there are two unique formulas which have several "extremal" properties with respect to all other formulas in $T(n, d)$. By proving that the E-optimal formula Q_n^{opt} is one of the two extremal formulas in $T(n, d_n)$, we can show various properties of E-optimal formulas.

The proofs of all theorems can be found in the supplements section of this issue.

2. E-Extremal Formulas. We call a (Chebyshev-type quadrature) formula $Q_n \in T(n, d)$ E-minimal in $T(n, d)$ and denote it by $Q_{n,d}^{\min}$ if

$$R_{n,d}^{\min}[p_{d+1}] = \min\{R_n[p_{d+1}] \mid Q_n \in T(n,d)\}. \tag{2.1}$$

Correspondingly, we define E-maximal formulas $Q_{n,d}^{\max} \in T(n, d)$ by

$$R_{n,d}^{\max}[p_{d+1}] = \max\{R_n[p_{d+1}] \mid Q_n \in T(n,d)\}. \tag{2.2}$$

Therefore, the following inequalities are valid for every $Q_n \in T(n, d)$:

$$Q_{n,d}^{\max}[p_{d+1}] \leqslant Q_n[p_{d+1}] \leqslant Q_{n,d}^{\min}[p_{d+1}].$$

Formulas with property (2.1) or (2.2) we call E-extremal. According to the arguments of Gautschi and Yanagiwara [9] for the existence of E-optimal formulas there exist E-minimal and E-maximal formulas in $T(n, d)$ for all d with

$$1 < d \leqslant d_n. \tag{2.3}$$

Remark. In the following we require for d the validity of (2.3), unless noted otherwise.

Our first result is the uniqueness of E-extremal formulas in $T(n, d)$.

THEOREM 1. *In $T(n,d)$ there exists only one E-minimal formula $Q_{n,d}^{\min}$ and only one E-maximal formula $Q_{n,d}^{\max}$.*

Definition (1.2) allows the possibility that some of the nodes coincide. It can be shown that E-extremal formulas have multiple nodes. Moreover, the two E-extremal formulas can be characterized by a special arrangement of these multiple nodes. To describe this arrangement we define for every Chebyshev-type quadrature formula Q_n the sequence $S(Q_n) := (s_i(Q_n))_{i=1}^{n-1}$ as follows:

$$s_i(Q_n) = \begin{cases} 0, & \text{if } x_{n+1-i} \neq x_{n-i}, \\ 1, & \text{if } x_{n+1-i} = x_{n-i} \text{ and } i \text{ odd}, \\ -1, & \text{if } x_{n+1-i} = x_{n-i} \text{ and } i \text{ even}. \end{cases} \tag{2.4}$$

We speak of a change of sign of the sequence $S(Q_n)$ (between $s_i(Q_n)$ and $s_{i+l}(Q_n)$) if

$$\begin{aligned} &\operatorname{sign}(s_i(Q_n)) = -\operatorname{sign}(s_{i+l}(Q_n)) \neq 0, \\ &s_{i+1}(Q_n) = s_{i+2}(Q_n) = \cdots = s_{i+l-1}(Q_n) = 0. \end{aligned} \tag{2.5}$$

THEOREM 2. *Let Q_n be E-extremal in $T(n,d)$. Then Q_n has at most d distinct nodes. Moreover,*

(i) *Let $d < n-1$. A formula Q_n is E-extremal in $T(n,d)$ if and only if $S(Q_n)$ has at least $(n-d-1)$ changes of sign. In this case the following holds: If the first nonzero term of $S(Q_n)$ is negative, then Q_n is E-minimal. If this term is positive, then Q_n is E-maximal. If $S(Q_n)$ has more than $(n-d-1)$ changes of sign, then Q_n is E-minimal as well as E-maximal and $T(n,d)$ contains only Q_n.*

(ii) *Let $d = n-1$. A formula Q_n is E-extremal in $T(n,d)$ if and only if $S(Q_n)$ has at least one nonzero term. If this term is negative, then Q_n is E-minimal. If this term is positive, then Q_n is E-maximal. If $S(Q_n)$ has at least one change of sign, then Q_n is E-minimal as well as E-maximal and $T(n,d)$ contains only Q_n.*

Theorem 2 shows that E-extremal formulas are interpolatory quadrature formulas (for definition see, e.g., [3]). The following theorem answers the question for other interpolatory quadrature formulas in $T(n,d)$ and for formulas with pairwise distinct nodes.

THEOREM 3. *Let $Q_{n,d}^{\min}$ and $Q_{n,d}^{\max}$ be the E-minimal and the E-maximal formula in $T(n,d)$. Let*

$$r \in \left(R_{n,d}^{\min}[p_{d+1}],\, R_{n,d}^{\max}[p_{d+1}]\right).$$

Then there exist formulas $\tilde{Q}_n$ and $\overline{Q}_n$ in $T(n,d)$ with $\tilde{R}_n[p_{d+1}] = \overline{R}_n[p_{d+1}] = r$ and

(i) *$\tilde{Q}_n$ has pairwise distinct nodes,*

(ii) *$\overline{Q}_n$ has at most $(d+1)$ distinct nodes.*

In the case of $d < n-1$, there exists for each such r even an infinite number of formulas with property (i). In the case of $d = n-1$ there exists for each such r only one formula $Q_n \in T(n, d)$ with $R_n[p_{d+1}] = r$ and this formula has pairwise distinct nodes.

A first justification for the consideration of E-extremal formulas is the fact that their first and their nth node have extremal properties with respect to all $Q_n \in T(n, d)$.

THEOREM 4. *Let x_i be the nodes of a formula $Q_n \in T(n, d)$, which is not E-extremal. Let $x_i^{\min}$ and $x_i^{\max}$ be the nodes of the E-extremal formulas $Q_{n,d}^{\min}$ and $Q_{n,d}^{\max}$ in $T(n, d)$. Then*

(i) $x_n^{\min} > x_n > x_n^{\max}$,

(ii) $(-1)^d x_1^{\min} > (-1)^d x_1 > (-1)^d x_1^{\max}$.

Therefore, it is also possible to characterize the E-minimal (E-maximal) formula in $T(n, d)$ to be that formula, whose nth node has the largest (smallest) value.

Furthermore, Theorem 4 may be helpful for the investigation of the question of whether all nodes of a formula $Q_n \in T(n, d)$ are contained in the interval $[a, b]$.

The formulas $Q_{n,d}^{\min}$ and $Q_{n,d}^{\max}$ are defined by the extremal property (2.1) and (2.2) of their remainder with respect to only one function, the monomial p_{d+1}. The following theorem shows that these extremal properties remain valid for a wide class of functions, which contains especially all monomials p_{d+2k-1} for all $k \in \mathbf{N}$.

THEOREM 5. *Let $Q_{n,d}^{\min}$ and $Q_{n,d}^{\max}$ be the E-minimal and the E-maximal formula in $T(n, d)$. Then, for all $f \in C^{d+1}$, $f^{(d+1)} \geqslant 0$, there hold*

(i) $R_{n,d}^{\min}[f] = \min\{R_n[f] \mid Q_n \in T(n, d)\}$,

(ii) $R_{n,d}^{\max}[f] = \max\{R_n[f] \mid Q_n \in T(n, d)\}$.

Another interpretation of Theorem 5 may be of interest:

Let K_{d+1} denote the Peano kernel of degree $d + 1$ ([7, p. 112], [3, p. 39]), of a formula Q_n in $T(n, d)$ and $K_{d+1}^{\min}$, resp. $K_{d+1}^{\max}$, the Peano kernels of the same degree of the E-extremal formulas $Q_n^{\min}$ or $Q_n^{\max}$ in $T(n, d)$. Theorem 5 implies the inequalities

$$K_{d+1}^{\min}(x) \leqslant K_{d+1}(x) \leqslant K_{d+1}^{\max}(x)$$

for all $x \in \mathbf{R}$.

3. E-Optimal Formulas. Our basic result for E-optimal formulas is given in the following theorem.

THEOREM 6. *Let $n \in \mathbf{N}$ and let Q_n^{opt} be E-optimal. Then Q_n^{opt} is E-extremal in $T(n, d_n)$.*

An E-optimal formula is therefore E-minimal or E-maximal in $T(n, d_n)$ and has the corresponding properties given in Section 2.

The first part of Theorem 2 has been proven for E-optimal formulas by Anderson and Gautschi [2]. The second part of Theorem 2 reduces all the remaining cases to only two formulas, characterized also by the value of the nth node according to

Theorem 4. Theorem 3 shows, in particular, the impossibility that different E-extremal formulas in $T(n, d_n)$ are both E-optimal. This answers the question of the uniqueness of E-optimal formulas [2], [7].

COROLLARY 1. *Let $n \in \mathbf{N}$. Then there exists one and only one E-optimal formula Q_n^{opt}.*

Therefore, by the result of Gautschi and Yanagiwara [9] mentioned above, it follows from Corollary 1 that

COROLLARY 2. *Let the weight function w be symmetric with respect to (a, b) and let $n \in \mathbf{N}$. Then d_n is odd and the E-optimal formula is symmetric.*

Rabinowitz and Richter [11] have shown that E-optimal rules minimize $|R_n|$ for formulas (1.2) in special function spaces. With the help of Theorem 5, a different justification for the consideration of E-optimal formulas is given by the following theorem.

THEOREM 7. *Let $n \in \mathbf{N}$ and Q_n^{opt} be the E-optimal formula and let $f \in C^{d_n+1}$, $f^{(d_n+1)} \geqslant 0$. If*

$$\operatorname{sign}\left(R_n^{\text{opt}}\left[p_{d_n+1}\right]\right) = \operatorname{sign}\left(R_n^{\text{opt}}[f]\right),$$

then

$$\left|R_n^{\text{opt}}[f]\right| = \min\left\{\left|R_n[f]\right| \mid Q_n \in T(n, d_n)\right\}.$$

4. Numerical Results for the Weight Function $w \equiv 1$. For the weight function $w \equiv 1$ there exist Chebyshev formulas in the strict sense for $n = 1, 2, \ldots, 7$ and $n = 9$—see, e.g., [7]. The E-optimal formulas have been computed by Gautschi and Yanagiwara [9] for $n = 8, 10, 11, 13$ and by Anderson and Gautschi [2] for $n = 12, 14, 15, 16, 17$. Anderson [1] has shown that these formulas, except for $n = 12$, are definite, i.e., there exists a representation of their remainder term of the form

$$R_n[f] = \frac{R_n\left[p_{d_n+1}\right]}{(d_n+1)!} f^{(d_n+1)}(\xi) \tag{4.1}$$

for every $f \in C^{d_n+1}$.

The present authors have computed the E-extremal formulas in $T(n, d_n)$ for $n \leqslant 25$ by a different method with the help of Theorem 2 resp. Theorem 3 [6]. The E-optimal formulas for $n = 18, \ldots, 25$ are given at the end of this section. These formulas are all definite. Theorem 5 implies that every $Q_n \in T(n, d_n)$ is also definite for $n \leqslant 25$, $n \neq 12$; for $n = 8, 10, 11, 13$ see Förster [4]. In case of definiteness, the comparison of the coefficients of $f^{(d_n+1)}(\xi)$ in (4.1) between the E-minimal and the E-maximal formula gives information as to how useful the choice of the E-optimal formula is in $T(n, d_n)$. These coefficients are listed in Table 1. In every case, the E-minimal formula is E-optimal. The numerical results correspond to the interval of integration $[-1, 1]$.

A conclusion of the above theorems is that the results of Gautschi and Monegato [8] and Förster [4] for $n = 8, 10, 11, 13$ remain valid for all $n \leqslant 25$, $n \neq 12$.

TABLE 1

n	d_n	$R^{min}_{n,d}[p_{d_n+1}]$	$R^{max}_{n,d}[p_{d_n+1}]$	$\frac{R^{max}_{n,d}[p_{d_n+1}]}{R^{min}_{n,d}[p_{d_n+1}]}$	
1	1	0.667 E 0	0.667 E 0	1	definite
2	3	0.178 E 0	0.178 E 0	1	definite
3	3	0.667 E-1	0.667 E-1	1	definite
4	5	0.339 E-1	0.339 E-1	1	definite
5	5	0.172 E-1	0.172 E-1	1	definite
6	7	0.102 E-1	0.102 E-1	1	definite
7	7	0.578 E-2	0.578 E-2	1	definite
8	7	0.2o2 E-2	0.541 E-2	2.68	definite
9	9	0.221 E-2	0.221 E-2	1	definite
10	9	0.119 E-2	0.153 E-2	1.29	definite
11	9	0.573 E-3	0.155 E-2	2.71	definite
12	9	0.663 E-4	0.121 E-2	18.25	R^{min}_{12} not definite R^{max}_{12} definite
13	11	0.384 E-3	0.440 E-3	1.15	definite
14	11	0.218 E-3	0.464 E-3	2.13	definite
15	11	0.102 E-3	0.384 E-3	3.76	definite
16	11	0.407 E-4	0.352 E-3	8.65	definite
17	13	0.105 E-3	0.117 E-3	1.11	definite
18	13	0.656 E-4	0.115 E-3	1.75	definite
19	13	0.399 E-4	0.108 E-3	2.71	definite
20	13	0.198 E-4	0.101 E-3	5.10	definite
21	13	0.613 E-5	0.860 E-4	14.03	definite
22	15	0.242 E-4	0.319 E-4	1.32	definite
23	15	0.159 E-4	0.273 E-4	1.72	definite
24	15	0.102 E-4	0.298 E-4	2.92	definite
25	15	0.594 E-5	0.262 E-4	4.41	definite

COROLLARY 3. *Let $n \leqslant 25$ and $w \equiv 1$. Let Q_n^{opt} be the E-optimal formula and $Q_n \in T(n, d_n)$.*

(a) *If in* (1.1) $b = -a$, *then for every* $m \in \mathbf{N}$,

$$0 \leqslant R_n^{opt}[p_m] \leqslant R_n[p_m].$$

(b) *If* $n \neq 12$ *and* $f \in C^{d_n+1}$, $f^{(d_n+1)} \geqslant 0$, *then*

$$0 \leqslant R_n^{opt}[f] \leqslant R_n[f].$$

Therefore, these E-optimal formulas satisfy also every optimality criterion of the form

$$\min\left\{ \sum_{i=d_n+1}^{\infty} a_i(R_n[p_i])^2 \middle| Q_n \in T(n, d_n) \right\}$$

with any $a_i \geqslant 0$ [7, p. 113]. They are, in particular for $n \neq 12$, also optimal in the sense of Sard [7, p. 112], [4].

The E-Optimal Formulas for $w \equiv 1 \qquad 18 \leqslant n \leqslant 25$

n = 18

$-x_1 =$	0.95611589370931681977	$= x_{18}$
$-x_2 = -x_3 =$	0.78339593833119703042	$= x_{16} = x_{17}$
$-x_4 =$	0.58679047283945639018	$= x_{15}$
$-x_5 = -x_6 =$	0.45756408008040941541	$= x_{13} = x_{14}$
$-x_7 =$	0.25737493728377540704	$= x_{12}$
$-x_8 = -x_9 =$	0.12068411927871514185	$= x_{10} = x_{11}$

n = 19

$-x_1 =$	0.95841522638659246454	$= x_{19}$
$-x_2 = -x_3 =$	0.79485226355878236323	$= x_{17} = x_{18}$
$-x_4 =$	0.60772484959475892451	$= x_{16}$
$-x_5 = -x_6 =$	0.48688511013054279206	$= x_{14} = x_{15}$
$-x_7 =$	0.29638895564058655907	$= x_{13}$
$-x_8 = -x_9 =$	0.16315108328419371742	$= x_{11} = x_{12}$
$-x_{10} =$	0.0	

n = 20

$-x_1 =$	0.96051482286129288228	$= x_{20}$
$-x_2 = -x_3 =$	0.80496515092537905967	$= x_{18} = x_{19}$
$-x_4 =$	0.63049631592920524269	$= x_{17}$
$-x_5 = -x_6 =$	0.50749481899047359478	$= x_{15} = x_{16}$
$-x_7 =$	0.35906562874648327105	$= x_{14}$
$-x_8 = -x_9 = -x_{10}$	0.15625951409613565727	$= x_{11} = x_{12} = x_{13}$

n = 21

$-x_1 =$	0.96243015157286074846	$= x_{21}$
$-x_2 = -x_3 =$	0.81403490074542027161	$= x_{19} = x_{20}$
$-x_4 =$	0.65167313907372323093	$= x_{18}$
$-x_5 = -x_6 =$	0.52555764207596964732	$= x_{16} = x_{17}$
$-x_7 =$	0.40559995128245393129	$= x_{15}$
$-x_8 = -x_9 = -x_{10} =$	0.18868995126113857640	$= x_{12} = x_{13} = x_{14}$
$-x_{11} =$	0.0	

(continues)

(continued)

$n = 22$

$-x_1 =$	0.96415710299556983171	$= x_{22}$
$-x_2 = -x_3 =$	0.82238727412825985167	$= x_{20} = x_{21}$
$-x_4 =$	0.66864696018494187221	$= x_{19}$
$-x_5 = -x_6 =$	0.54600146439908270396	$= x_{17} = x_{18}$
$-x_7 =$	0.43289922578951637757	$= x_{16}$
$-x_8 = -x_9 =$	0.24362435512429351622	$= x_{14} = x_{15}$
$-x_{10} =$	0.18706881618423297318	$= x_{13}$
$-x_{11} =$	0.0	$= x_{12}$

$n = 23$

$-x_1 =$	0.96570343338357257096	$= x_{23}$
$-x_2 = -x_3 =$	0.83018849753913168834	$= x_{21} = x_{22}$
$-x_4 =$	0.68178827221824105045	$= x_{20}$
$-x_5 = -x_6 =$	0.56773078130524428871	$= x_{18} = x_{19}$
$-x_7 =$	0.45254730818175202350	$= x_{17}$
$-x_8 = -x_9 =$	0.26810913164371012869	$= x_{15} = x_{16}$
$-x_{10} =$	0.25355181302970919482	$= x_{14}$
$-x_{11} = -x_{12} =$	0.0	$= x_{13}$

$n = 24$

$-x_1 =$	0.96712730714333769553	$= x_{24}$
$-x_2 = -x_3 =$	0.83729311756137103729	$= x_{22} = x_{23}$
$-x_4 =$	0.69467063974654513014	$= x_{21}$
$-x_5 = -x_6 =$	0.58616217620434405885	$= x_{19} = x_{20}$
$-x_7 =$	0.47487624160088429065	$= x_{18}$
$-x_8 = -x_9 = -x_{10} =$	0.29353907385470281834	$= x_{15} = x_{16} = x_{17}$
$-x_{11} =$	0.09382293173785193807	$= x_{14}$
$-x_{12} =$	0.0	$= x_{13}$

$n = 25$

$-x_1 =$	0.96844773854353010676	$= x_{25}$
$-x_2 = -x_3 =$	0.84375871505247479493	$= x_{23} = x_{24}$
$-x_4 =$	0.70773522849837207585	$= x_{22}$
$-x_5 = -x_6 =$	0.60110284438058970914	$= x_{20} = x_{21}$
$-x_7 =$	0.50135993977793685911	$= x_{19}$
$-x_8 = -x_9 = -x_{10} =$	0.31836145542090472915	$= x_{16} = x_{17} = x_{18}$
$-x_{11} =$	0.16110820932771201152	$= x_{15}$
$-x_{12} = -x_{13} =$	0.0	$= x_{14}$

5. Examples. Table 1 shows that in case of $w \equiv 1$ the sets $T(n, d_n)$ for $n \leqslant 25$ and $d_n < n$ contain an infinite number of elements (see Theorem 3). The same is true for the examples computed by Anderson and Gautschi [2] in case of other weight functions. The following example shows with the help of Theorem 2 the possibility that for $d_n < n$ the set $T(n, d_n)$ contains only one element.

Let the weight function w be given by $w(x) := \sqrt{1 - x^2}$. The corresponding Gauss-formula G_5 with 5 nodes and therefore degree 9 is given by (see Szegö [12, p. 344])

$$G_5[f] = \frac{\pi}{24}\left\{f\left(-\frac{\sqrt{3}}{2}\right) + 3f\left(-\frac{1}{2}\right) + 4f(0) + 3f\left(\frac{1}{2}\right) + f\left(\frac{\sqrt{3}}{2}\right)\right\}.$$

Because of $m_0 = \pi/2$ the formula G_5 is a Chebyshev-type quadrature formula (1.2) with the twelve nodes

$$x_1 = -\frac{\sqrt{3}}{2} = -x_{12},$$

$$x_2 = x_3 = x_4 = -\frac{1}{2} = -x_{11} = -x_{10} = -x_9,$$

$$x_5 = x_6 = 0 = -x_8 = -x_7.$$

So G_5 is an element of $T(12,9)$. By (2.4) the sequence $S(G_5)$ is given by

$$S(G_5) = (0,1,-1,0,-1,1,-1,0,-1,1,0)$$

and has four changes of sign; see (2.5). Theorem 2(i) shows that $T(12,9)$ contains only the element G_5. Furthermore, G_5 is also the only element of $T(12,8)$, and G_5 is the E-maximal formula $Q_{12,7}^{\max}$ in $T(12,7)$.

In the case of $w \equiv 1$ and $n \leqslant 25$ the nodes of the E-minimal formula $Q_{n,d_n}^{\min}$ are contained in the interval $(-1,1)$. Therefore, in these cases, using Theorem 4, the nodes of every formula $Q_n \in T(n, d_n)$ are also contained in $(-1,1)$. But this is not so, in general, for every weight function w and every $n \in \mathbf{N}$. The following example shows that there exist even Chebyshev quadrature formulas in the strict sense, i.e., $d_n \geqslant n$, with nodes not all contained in $[a,b]$:

Let w be a weight function on $(-1,1)$ with $w(x) := (1-x^2)^{-4/5}$. A simple calculation with the help of Newton's identities (see [7, p. 104]) shows that for $n = 3$, 4, 6, 7 the Chebyshev quadrature formulas in the strict sense exist and that their first and last nodes are not contained in $[-1,1]$.

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